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1995 J. Phys. A: Math. Gen. 28 3525

(http://iopscience.iop.org/0305-4470/28/12/023)

J. Phys. A: Math. Gen. 28 (1995) 3525–3532. Printed in the UK

Some observations on Darboux's theorem, isospectral Hamiltonians and supersymmetry

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Received 23 June 1994, in final form 2 September 1994

Abstract. A method to generate isospectral supersymmetric Hamiltonians is given which is similar but inequivalent to the usual approach based on Darboux's theorem. Explicit calculations are made for the potential $V(r) = -2\,\mathrm{sech}^2 r$ and the conditionally exactly solvable potential given by $V(r) = Ar^{2/3} + B/r^{2/3} - G_0/r^2$.

In recent years isospectral Hamiltonians, i.e. Hamiltonians with the same eigenvalues, have been studied by several authors [1–5]. Among various approaches to general isospectral Hamiltonians, the most popular one is based on a theorem about second-order linear differential equations first discussed by Darboux [6] over a century ago.

Abraham and Moses [1] have developed a formalism to generate isospectral Hamiltonians based on the Gel'fand–Levitan equation [7]. Luban and Pursey [2] showed that the approach proposed by Abraham and Moses and the one based on Darboux's theorem are inequivalent. Recently Schnizer and Leeb [8] have generalized Darboux transformations to find exactly solvable Schrödinger equations. Supersymmetric quantum mechanics [9–13] (SSQM) which has been the subject of much attention over the past decade or so provides another approach which gives two isospectral Hamiltonians which are called the bosonic and fermionic parts of the supersymmetric system.

In this paper we show that the usual Darboux theorem when applied to a supersymmetric Hamiltonian leads to its supersymmetric partner which is isospectral with the original Hamiltonian; that repeated application of Darboux's theorem does not give any more isospectral supersymmetric Hamiltonians. However, it is shown that a modified approach can generate a series of exactly solvable supersymmetric Hamiltonians if one is given a single exactly solvable supersymmetric Hamiltonian. The Hamiltonians are different from those obtained by Schnizer and Leeb [8]. Before we discuss the connection between Darboux's theorem and supersymmetry let us briefly present the essential details of SSQM.

The Hamiltonian of SSQM is given by

$$H = \frac{p^2}{2} + \frac{1}{2}W^2(x) - \frac{[\Psi^*, \Psi]}{2}W'(x) \tag{1}$$

where the Ψ satisfy the anticommutation relation

$$\{\Psi^*, \Psi\} = 1. \tag{2}$$

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One can use

$$\Psi^* = \sigma^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$
$$\Psi = \sigma^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{3}$$

The form of Ψ and Ψ^* given in (3) indicates that H given in (1) is a 2×2 matrix. In fact we can write

$$H = \begin{bmatrix} H_+ & 0 \\ 0 & H_- \end{bmatrix}. \tag{4}$$

Since

$$[\Psi^*, \Psi] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5}$$

H_+, H_- are given by

$$H_+ = \frac{p^2}{2} + \frac{1}{2}W^2(x) + \frac{W'(x)}{2} \tag{6}$$

$$H_- = \frac{p^2}{2} + \frac{1}{2}W^2(x) - \frac{W'(x)}{2}. \tag{7}$$

The eigenvalue equation can then be written as

$$H\varphi = E\varphi \tag{8}$$

where φ is a two-component vector given by

$$\varphi = \begin{bmatrix} \varphi^+ \\ \varphi^- \end{bmatrix}. \tag{8a}$$

From (8) the equations for φ^+, φ^- are found to be

$$\frac{\mathrm{d}^2\varphi_+}{\mathrm{d}x^2} + (2E^+ - W^2(x) - W'(x))\varphi_+(x) = 0 \tag{9}$$

and

$$\frac{\mathrm{d}^2\varphi_-}{\mathrm{d}x^2} + (2E^- - W^2(x) + W'(x))\varphi_-(x) = 0. \tag{10}$$

The Hamiltonians corresponding to (9) and (10), i.e.

$$H = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - (W^2 \pm W') \tag{11}$$

are isospectral. (Here $W' = \mathrm{d}w/\mathrm{d}x$.)

Now let us consider the Schrödinger equation

$$\frac{d^2\Psi}{dr^2} + (V_0(r) - E)\Psi = 0. \tag{12}$$

Here $V_0(r)$ may include the angular momentum term $\ell(\ell+1)/r^2$ and also the term $\theta^2 h(r)$ discussed by Schnizer and Leeb for this would make no difference to our argument. Let $\zeta_0(\alpha, r)$ be a solution of (1). If $\eta_0(\beta, r)$ is another solution of (1) then, according to Darboux's result, as formulated by Schnizer and Leeb, the function η_1 defined by

$$\eta_1 = \frac{W(\eta_2(\beta, r) - \zeta_0(\alpha, r))}{\zeta_0(\alpha, r)} \tag{13}$$

where $W(\Psi_2, \Psi_2)$ is the Wronskian given by

$$W(\Psi_1, \Psi_2) = \Psi_1 \frac{d\Psi_2}{dr} - \frac{d\Psi_1}{dr}\Psi_2 \tag{14}$$

satisfies a Schrödinger equation with potential

$$V_1(r) = V_0(r) - 2\frac{d}{dr}(\zeta_0'/\zeta_0). \tag{15}$$

To see the connection with supersymmetry let

$$\zeta_0(\alpha, r) = \exp\left(\int w_0 \, dr\right). \tag{16}$$

$\zeta_0(r)$ satisfies the Schrödinger equation

$$\zeta_0''(r) - (w_0^2 + w_0')\zeta_0 = 0 \tag{17}$$

or

$$\zeta_0''(r) - (V_0(r) - E)\zeta_0 = 0 \tag{18}$$

where

$$V_0(r) - E = w_0^2 + w_0'. \tag{19}$$

Hence ζ_0 corresponds to the ground-state solution of a supersymmetric Hamiltonian with super potential w_0. Now,

$$w_0 = \frac{\zeta_0'(\alpha, r)}{\zeta_0(\alpha, r)}. \tag{20}$$

Therefore, from (15)

$$V_1(r) = V_0(r) - 2w_0' = w_0^2 - w_0' \tag{21}$$

which is the superpartner of (19). Thus Darboux's theorem implies the well known property of SUSYQM. If we apply Darboux's theorem again to (21) we will get back (19). However, if we use a modified form of Darboux's theorem a series of new exactly solvable Hamiltonians can be generated. Let $\eta_0(r)$, $\eta_1(r)$ be two independent solutions of (9) (eventually only one of them can be normalized). Take the solutions η_0, η_1 in the following form:

$$\eta_0(r) = \exp\left(\int w_0(r)\,\mathrm{d}r\right) \tag{22}$$

and

$$\eta_1(r) = \exp\left(\int w_1(r)\,\mathrm{d}r\right) \tag{23}$$

such that

$$w_0^2 + w_0' = w_1^2 + w_1'. \tag{24}$$

To solve (23) we put

$$W_1(r) = w_0(r) + u(r) \tag{25}$$

whence u satisfies the Riccati equation

$$u' + 2uw_0 + u^2 = 0. \tag{26}$$

Putting $u = 1/f$, equation (26) can be written as

$$f' - 2fw_0 = 1. \tag{27}$$

Equation (27) is a first-order linear differential equation in f. Its solution is given by

$$f = \exp\left\{2\int^r w_0(t)\,\mathrm{d}t\right\}\int^r \mathrm{d}r\, \exp\left\{-\int^r 2w_0(t)\,\mathrm{d}t\right\} + k\exp\left\{2\int^r w_0(t)\,\mathrm{d}t\right\}. \tag{28}$$

Therefore

$$u = \frac{1}{f} = \frac{\exp\{-2\int^r w_0(t)\,\mathrm{d}t\}}{k + \int^r \mathrm{d}r\, \exp\{-\int^r 2w_0(t)\,\mathrm{d}t\}} \tag{29}$$

where k is an integral constant.

Now define,

$$\bar{\eta} = \frac{W(\eta_0, \eta_1)}{\eta_1(p(r))^m} \tag{30}$$

where $p(r)$ is any function of r and is not necessarily $h(r)$ even if one includes $\theta^2 h(r)$ in the definition of $V_0(r)$. Also $W(\eta_0, \eta_1)$ is the Wronskian defined by

$$W(\eta_0, \eta_1) = \eta_0(r)\frac{\mathrm{d}\eta_1(r)}{\mathrm{d}r} - \frac{\mathrm{d}\eta_0(r)}{\mathrm{d}r}\eta_1(r) \tag{31}$$

and m is any real number. ($m = 1/2$ gives the form taken by Schnizer and Leeb [8].) Note that the Wronskian in (30) is trivial in the present case but we keep it in this form to show the apparent similarity between our results and those of Schnizer and Leeb [8]. Now $\bar{\eta}$ can be written as

$$\bar{\eta} = \frac{(w_1 - w_0)\eta_0}{p^m} \qquad (p = p(r)). \tag{32}$$

Therefore

$$\bar{\eta}' = \frac{\mathrm{d}\bar{\eta}}{\mathrm{d}r} = \bar{\eta}\left[w_0 + \frac{(w_0' - w_1')}{w_0 - w_1} - \frac{mp'}{p}\right]. \tag{33}$$

Differentiating again we get, after some simplification,

$$\bar{\eta}'' = \bar{\eta}\left[w_0^2 + w_0' - 2w_1' + \frac{2mw_1 p'}{p} + \frac{m(m+1)p'^2}{p^2} - \frac{mp''}{p}\right] \tag{34}$$

where we have used the result

$$w_0^2 + w_0' = w_1^2 + w_1' = V_0(r) - E. \tag{35}$$

Hence $\bar{\eta}$ satisfies a Schrödinger equation with potential $V_1(r)$ given by

$$V_1(r) = V_0(r) - 2(p(r))^m \frac{\mathrm{d}}{\mathrm{d}r}(p(r))^{-m}\frac{\mathrm{d}}{\mathrm{d}r}\ell n[(p(r))^{m/2}\eta_1(r)] \tag{36}$$

for $p(r) = 1$

$$V_1(r) = w_1^2 - w_1' \tag{37}$$

which gives a new Hamiltonian which is isospectral with the SUSY Hamiltonian with superpotential w_0. (Note that $w_1^2 - w_1'$ is different from $w_0^2 - w_0'$.) If we apply the same procedure to the potential $V_1(r)$ given in (36) and define

$$\bar{\eta}_2 = \frac{W(\bar{\eta}, \bar{\eta}_1)}{\bar{\eta}_1(p(r))^m} \tag{38}$$

where $\bar{\eta}, \bar{\eta}_1$ are independent solutions of the Schrödinger equation

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}r^2} + (V_1(r) - E)\Psi = 0 \tag{39}$$

then $\bar{\eta}_1$ satisfies a Schrödinger equation with potential $V_2(r)$, given by

$$V_2(r) = V_0(r) - 2(p(r))^m \frac{\mathrm{d}}{\mathrm{d}r}\left[(p(r))^{-m}\frac{\mathrm{d}}{\mathrm{d}r}\ell n(\bar{\eta}_1/\bar{\eta}_2)\right] \tag{40}$$

if we take $p(r) = 1$, then

$$V_2(r) = w_2^2 + w_2' \tag{41}$$

where w_2 satisfies the Riccati-like equation

$$w_2^2 - w_2' = w_1^2 - w_1'. \tag{42}$$

Because of (42), w_2 is isospectral with (19). If we write

$$w_2(r) = w_1(r) + g(r) \tag{43}$$

then $g(r)$ is given by

$$g(r) = \frac{\exp\{\int^r 2w_1(t)\,\mathrm{d}t\}}{k' + \int^r \mathrm{d}r\ \exp\{\int^r 2w_1(t)\,\mathrm{d}t\}} \tag{44}$$

where k' is an integration constant. In the following we give a simple example to show how one can explicitly obtain the isospectral Hamiltonians. Let us take $w_0 = 2\tanh r$, $p(r) = 1$, then the corresponding potential is

$$V(r) = -2\,\mathrm{sech}^2\, r. \tag{45}$$

This is isospectral with a Hamiltonian with potential

$$V_1(r) = w_1^2 - w_1' \tag{46}$$

where

$$\begin{aligned} w_1 &= 2\tanh r + u(r) \\ &= 2\tanh r + \frac{\exp\{-2\int^r w(r)\,\mathrm{d}r\}}{k + \int^r \exp\{-2\int w(r)\,\mathrm{d}r\}\,\mathrm{d}r} \\ &= 2\tanh r + \frac{3\,\mathrm{sech}^4\, r}{k' + \tanh r(\mathrm{sech}^2\, r + 2)} \qquad (k' = 3k). \end{aligned} \tag{47}$$

If one takes a non-constant $p(r)$ a different potential results. For example $(p(r))^m = 1/r$ gives

$$V_1(r) = V_0(r) - \frac{2}{r}w_1 - 2w_1' \tag{48}$$

where $w_1(r)$ is given by (47). If we apply the same procedure to the potential (46), we get a Hamiltonian isospectral with (46) and with a superpotential w_2 given by

$$w_2 = w_0 + u(r) + g(r) \tag{49}$$

where $w_0 + u(r)(= w_1(r))$ is given by (47) and

$$\begin{aligned} g(r) &= \frac{\exp\{2\int^r w_1(r)\,\mathrm{d}r\}}{k'' + \int \exp\{2\int^r w_1(r)\,\mathrm{d}r\}\,\mathrm{d}r} \\ &= I_1(r)/I_2(r) \end{aligned} \tag{50}$$

where

$$I_1(r) = \cosh^4 r(k' + \tanh r(\mathrm{sech}^2\, r + 2))^2 \tag{51}$$

and

$$I_2(r) = r(\tfrac{3}{8}k'^2 - \tfrac{3}{2}) + \sinh 2r(\tfrac{1}{4}k'^2 + 1) + \sinh r(\tfrac{1}{32}k'^2 + \tfrac{1}{8})$$
$$+ k'(\cosh^4 r + \tfrac{1}{2}\cosh 2r) - \tanh r + k'' \qquad (52)$$

where k' and k'' are integrating constants.

Recently Dutra [14] has found exact solutions of a class of potentials called conditionally exactly solvable potentials. Since these class of potentials can be put exactly in the supersymmetric form [15], our formalism can also be applied to these classes of potentials. For example, if we take the superpotential to be

$$W(r) = ar^{1/3} + \frac{b}{r} + \frac{d}{r^{1/3}} \qquad (53)$$

we get one of the potentials found by Dutra [14, 15] (the so-called conditionally exactly solvable potential), i.e.

$$V(r) = ar^{2/3} + \frac{B}{r^{2/3}} - \frac{5/36}{r^2} \qquad (54)$$

where $A = a^2$ and $B = d^2 + 2bd$. If we apply the method of obtaining isospectral Hamiltonians described above then the Hamiltonian with a potential given by (54) is isospectral with a Hamiltonian with a potential

$$V_1(r) = V(r) - 2\frac{dw_1}{dr} \qquad (56)$$

where

$$w_1(r) = ar^{1/3} + \frac{1/6}{r} + \frac{d}{r^{1/3}} + \frac{\bar{I}_1(r)}{\bar{I}_2(r)} \qquad (57)$$

$$\bar{I}_1(r) = r^{-1/3}e^{-(3/2)r^{2/3}}(ar^{2/3} + 2d) \qquad (58)$$

and

$$\bar{I}_2(r) = \sqrt{\frac{3\pi}{8a}}e^{3d^2/2a}\,\mathrm{erf}\left[\frac{3}{2}ar^{2/3} + \frac{3}{2a}d\right] + k \qquad (59)$$

where k is an integrating constant and erf(z) is the error function [16] defined by

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt. \qquad (60)$$

To conclude, a method to generate isospectral supersymmetric Hamiltonians is given which is similar but inequivalent to Darboux's method.

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